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LETTER TO THE EDITOR

Critical isotherm of Dyson's hierarchical model

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Abstract. The critical isotherm for Dyson's hierarchical model in one dimension with potential decaying as $r^{-(1+\sigma)}$ is investigated in the classical regime $0<\sigma\leqslant\frac{1}{2}$, corresponding to a Gaussian fixed point of the exact renormalization group equation. Exact series expansions for the magnetization at the critical point are developed and analysed. We conclude that $\delta=3\cdot0\pm0\cdot05$ for $0<\sigma\leqslant\frac{1}{2}$, confirming the belief that *all* exponents take on their classical values in the Gaussian, renormalization group fixed point, regime.

In the renormalization group (RG) approach to critical phenomena there are certain ranges of physical parameters where the appropriate (physical) fixed point of the RG equation is a simple Gaussian. Two examples are the dimensionality d greater than 4 for short-range interactions (Wilson and Kogut 1974) and σ less than $d/2$ for long-range interactions that decay as $r^{-(\sigma+d)}$ as $r\to\infty$ (Fisher *et al* 1972, Bleher and Sinai 1974). In these 'Gaussian regimes' it is generally believed that critical exponents take on their classical values. Renormalization group arguments confirm this belief for various exponents (γ, ν, etc). At the critical or fixed point, however, in these 'classical regimes' the free energy corresponding to the fixed point Hamiltonian does not exist and consequently, the usual RG prescription for the critical exponent δ, among others, is no longer valid. Nevertheless it is generally believed that δ, in particular, sticks at its classical value of 3 in the classical regimes. A plausible argument for δ being 3 in the classical regime has been given by Ma (1976), when the Hamiltonian is of Ginzberg–Landau form, but as part of his argument he uses the explicit solution of the magnetization in the Landau or classical approximation. It is therefore of some importance to investigate the exponent δ in various classical regimes.

In this letter we determine δ numerically for Dyson's hierarchical model ($d=1$) by series expansion methods.

The L-level, 2^L-spin hierarchical model has Hamiltonian (Dyson 1969)

$$\mathscr{H}_L=-\sum_{p=1}^{L}2^{-(1+\sigma)p}\sum_{r=1}^{2L-p}S_{p,r}^2-HS_{N,1} \tag{1}$$

where

$$S_{p,r}=S_{p-1,2r-1}+S_{p-1,2r},\qquad r=1,2,\ldots,2^{L-p};\qquad p=1,2,\ldots,L$$

and $S_{0,r}=\pm1$. The potential for this model decays essentially as $r^{-(\sigma+1)}$. Unlike the corresponding power law potential Ising model, however, the RG can be realized exactly for the hierarchical model and it can be shown rigorously that the classical or Gaussian regime corresponds to $0<\sigma\leq\frac{1}{2}$ (Bleher and Sinai 1973).

To derive the magnetization series expansion in the variable

$$z=\exp(-2\beta H) \tag{2}$$

we utilize the exact recursion relation for the partition function (Kim and Thompson 1977)

$$Q_L(\beta, H;\sigma)=\pi^{-\frac{1}{2}}\int_{-\infty}^{\infty}\exp(-x^2)(Q_{L-1}(\beta, H+2^{1-\frac{1}{2}(1+\sigma)L}\beta^{-\frac{1}{2}}x;\sigma))^2\,\mathrm{d}x \tag{3}$$

where

$$Q_L(\beta, H;\sigma)=\sum_{\{S_{0,r}=\pm1\}}\exp(-\beta\mathscr{H}_L) \tag{4}$$

and

$$Q_0(\beta, H;\sigma)=z^{-\frac{1}{2}}(1+z). \tag{5}$$

A scaling -RG argument applied to (1), which in essence assumes that the limiting fixed point free energy exists, gives (Kim and Thompson 1977)

$$\delta=(1+\sigma)/(1-\sigma). \tag{6}$$

For reasons stated above it is likely, however, that this formula does not hold for $0<\sigma\leq\frac{1}{2}$, leaving open the possibility of (6) being valid for $\frac{1}{2}\leq\sigma<1$ and δ sticking at 3 for $0<\sigma\leq\frac{1}{2}$.

Using (3) and (5) and the known critical temperatures of (1) obtained by Kim and Thompson (1977), the magnetization series of the 2^L-spin system *at* the critical temperature:

$$M_L(z;\sigma)=\frac{\partial}{\partial(\beta H)}2^{-L}\ln Q_L(\beta_c(\sigma), H;\sigma)=1-\sum_{n=1}^{\infty}a_{n,L}(\sigma)z^n \tag{7}$$

can easily be developed recursively on a computer for various L. We then require $a_n=\lim_{L\to\infty}a_{n,L}$. The exact analytic form of $a_{n,L}$ for $n=1$ and 2 behaves for large L as

$$a_{n,L}=a_n(1+c_12^{-\sigma L}+c_22^{-2\sigma L}+\ldots+d_12^{-(1+\sigma)L}+d_22^{-2(1+\sigma)L}+\ldots). \tag{8}$$

Assuming that (8) holds for all n we can then obtain accurate values for a_n by appropriate extrapolation of the $a_{n,L}$ sequences.

Even though the computer generation of the $a_{n,L}$ is simple and exact, the loss of significant digits in the iteration of the recurrence relations and in the process of taking the logarithm (equation (7)) causes difficulties. In order to overcome this numerical instability the calculations were carried out with a precision of 200 *decimal* digits using Brent's (Brent 1977) multiple precision FORTRAN package. This enabled us to obtain a reasonable number of terms for the series (7). In particular, we obtained a_n up to $n=8$, 10 and 12 for $\sigma=0{\cdot}05$, $0{\cdot}15$ and $0{\cdot}25$ respectively.

Our analysis of the magnetization series on the critical isotherm is based on a simple method first used by Gaunt (1967). Thus, assuming that to leading order $M(z;\sigma)=\lim_{L\to\infty}M_L(z;\sigma)$ behaves as

$$M(z;\sigma)\sim(1-z)^{1/\delta}\qquad(\beta=\beta_c;\ z\to1-) \tag{9}$$

it follows that

$$-\frac{\mathrm{d}\ln M}{\mathrm{d}\ln z} \equiv \sum_{n=1}^{\infty} (\delta_n)^{-1} z^n \sim \frac{z}{\delta(1-z)} = \frac{z}{\delta} + \frac{z^2}{\delta} + \ldots. \tag{10}$$

That is, the reciprocals of the series coefficients δ_n obtained from the a_n, should themselves approach δ. In tables 1, 2 and 3 we have shown the coefficients δ_n and their linear and quadratic extrapolants, corresponding to the first two columns of the Neville table (see e.g. Gaunt and Guttmann 1974), for three representative values of σ.

From the tables we estimate that $\delta = 3{\cdot}0 \pm 0{\cdot}05$ for all cases shown, which strongly suggests that δ is indeed 3 for all σ in the classical regime $0 < \sigma \leq \frac{1}{2}$. For the sake of comparison the formula (6) for δ gives $\delta = 1{\cdot}105$, $1{\cdot}353$, and $1{\cdot}667$, respectively, for the values $\sigma = 0{\cdot}05$, $0{\cdot}15$ and $0{\cdot}25$.

We might add that critical exponents other than δ for the 'asymptotic hierarchical model' of Bleher and Sinai (1973), namely $\beta = \frac{1}{2}$, $\gamma = \gamma' = 1$, $\eta = 2 - \sigma$ have been obtained rigorously in the classical regime $0 < \sigma \leq \frac{1}{2}$. These results, together with the RG results $\nu = 1/\sigma$ and $\alpha_s = 2 - 1/\sigma$ ($\alpha = 0$) on the high temperature side, and the most probable value $\delta = 3$ at the critical point, completes the picture for the hierarchical model in the classical regime.

Table 1. $\sigma = 0{\cdot}05$ ($\beta_c = 0{\cdot}018\,276\,30$).

n	δ_n	Linear extrapolants	Quadratic extrapolants
1	3·7121	0	0
2	3·4213	3·1306	0
3	3·3130	3·0962	3·0790
4	3·2544	3·0785	3·0609
5	3·2170	3·0674	3·0506
6	3·1907	3·0595	3·0437
7	3·1712	3·0542	3·0410
8	3·1561	3·0503	3·0387

Table 2. $\sigma = 0{\cdot}15$ ($\beta_c = 0{\cdot}061\,458\,12$).

n	δ_n	Linear extrapolants	Quadratic extrapolants
1	3·8529	0	0
2	3·5048	3·1568	0
3	3·3766	3·1202	3·1020
4	3·3064	3·0956	3·0709
5	3·2614	3·0818	3·0610
6	3·2298	3·0716	3·0514
7	3·2062	3·0642	3·0457
8	3·1877	3·0584	3·0411
9	3·1726	3·0538	3·0376
10	3·1605	3·0500	3·0347

Table 3. $\sigma = 0{\cdot}25$ ($\beta_c = 0{\cdot}116\,028\,976\,5$).

n	δ_n	Linear extrapolants	Quadratic extrapolants
1	4·1500	0	0
2	3·7009	3·2519	0
3	3·5442	3·2308	3·2203
4	3·4507	3·1703	3·1098
5	3·3926	3·1601	3·1447
6	3·3500	3·1367	3·0898
7	3·3178	3·1249	3·0956
8	3·2924	3·1149	3·0847
9	3·2718	3·1065	3·0770
10	3·2545	3·0992	3·0702
11	3·2399	3·0933	3·0665
12	3·2272	3·0881	3·0626

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References

Bleher P M and Sinai Ja G 1973 *Commun. Math. Phys.* **33** 23–42
—— 1974 *Sov. Phys.-JETP* **40** 195–7
Brent R P 1977 *Trans. Assoc. Comput. Mach.–Software* to be published
Dyson F J 1969 *Commun. Math. Phys.* **12** 91–107
Fisher M E, Ma S-K and Nickel B G 1972 *Phys. Rev. Lett.* **29** 917–20
Gaunt D S 1967 *Proc. Phys. Soc.* **92** 150–8
Gaunt D S and Guttmann A J 1974 *Phase Transitions and Critical Phenomena* vol. 3, eds C Domb and M S Green (New York: Academic)
Kim D and Thompson C J 1977 *J. Phys. A: Math. Gen.* **10** in the press
Ma S-K 1976 *Modern Theory of Critical Phenomena* (New York: Benjamin)
Wilson K G and Kogut J 1974 *Phys. Rep.* **12** 75–200